2.34.1. More Duality Problems

A. For each of the following sentences, decide if that sentence is **true** or **false**.

1. If two sentences are connective duals, they're also semantic duals.

2. If two sentences are semantic duals, they're also connective duals.

3. If one sentence is the Tilde Insertion dual of a second, then the two sentences are semantic duals.

4. If a sentence is a negation, then it's a (connective and semantic) self-dual.

5. If the only connective(s) in a sentence are tilde(s), then that sentence is a (connective and semantic) self-dual.

B. We noted that if Sentence 2 follows from Sentence 1, the 'following from' relation is preserved under duality only if we swap the order of the sentences.

> If Sentence 2 follows validly from Sentence 1, then the dual of Sentence 1 follows validly from the dual of Sentence 2.

But another way of putting this claim, without swapping the sentences, is to replace the first "**follows from**" with "**entails**".[1]

> If Sentence 1 **entails** Sentence 2, then Dual of Sentence 1 **follows validly from** the dual of Sentence 2.

For example, "~(P ∨ Q)" entails "~P"; and "~(P ∧ Q)" follows validly from "~P".

That suggests that '**entails**' **and** '**follows validly from**' can themselves paired as **duals**, to be swapped under duality.

[1] Recall (from 1.7, note 1) that if Sentence 1 **entails** Sentence 2, then Sentence 2 **follows validly from** Sentence 1.

Now, a core feature of duality is that it allows us to capture two different logical laws by swapping out dual pairs in one law to get the other – as in the following example.

	conjunction		**true**		**true**
A		is only		when both its parts are	
	disjunction		**false**		**false**

Treating ‘**entails**’ and ‘**follows validly from**’ as dual relations, build the dual of each of the following logical claims by swapping out duals.

1. Any sentence **entails** a **tautology**.

2. Only a **tautology follows validly from** its own **negation**.

3. A **conjunction entails** each of its parts.

4. A sentence **entails** the **negation** of its **negation**.

5. If a **disjunction** is a **tautology**, then the **negation** of one part of the **disjunction entails** the other part.[2]

Are (1) through (5) true? Are their duals true?

C. **(1)** Build a truth table for the sentence “$((P \wedge Q) \vee (P \wedge {\sim}Q))$” and for its connective dual. Do the two sentences have the truth tables we’d expect from True/False Swap duality?

(2) Based on your answers to **(1)**, what sentence is the **dual of a sentence letter** such as “P”, according to the True/False Swap? Does that dual truth table agree with the sentence you get from performing the Connective Swap on “P”? What is the Tilde Insertion dual of “P”? Does the Tilde Insertion dual of “P” take same the truth table as the Connective Swap dual of “P”?

[2] See 2.42. for further discussion.

D. Which of the following formal languages allow us to build the **connective dual** of every sentence in that language? Which allow us to build the **Tilde Insertion dual** of every sentence in the language? Which allow us to build **a semantic dual** sentence for every sentence in the language?

$\{\wedge, \vee\}$	$\{\vee, \sim\}$
$\{\sim\}$	$\{\vee\}$

E. Recall that we earlier[3] categorized arguments (with premises conjoined into one sentence) according to the status of the premise and the conclusion.

C/: The premise is a contradiction.
N/: The premise is neither a contradiction nor a tautology.
T/: The premise is a tautology.

/C: The conclusion is a contradiction.
/N: The conclusion is neither a contradiction nor a tautology.
/T: The conclusion is a tautology.

The result was a set of nine classes of arguments (where, e.g., **C/T** had a contradictory premise and tautological conclusion).

C/C	**C/N**	**C/T**
N/C	**N/N**	**N/T**
T/C	**T/N**	**T/T**

And we found that in every class except N/N, the class an argument is in is sufficient to know whether the argument is valid or invalid. So, for example, **every C/ argument is bound to be valid** (regardless of the status of its conclusion).

But with duality of tautology, contradiction, and argument now in hand, we can extend duality to these nine classes as well. For the dual of a given argument we (i) switch premise and conclusion, and (ii) replace each such sentence by its connective dual. And **dual of a tautology is a contradiction (and vice versa)**, while **dual of a neither-tautology-nor-contradiction is another 'neither'**. So:

[3] In 2.19 § 2.

the dual of the **N/C** argument class is **T/N**; and the dual of the **T/C** argument class is **T/C**. (**T/C** is a self-dual class.)

1. Determine the **dual** of each of the nine argument classes. State which are **self-duals**.

C/C	**C/N**	**C/T**
N/C	**N/N**	**N/T**
T/C	**T/N**	**T/T**

2. For each of the following sentences, state the **dual** sentence. (Fill in the blanks for incomplete sentences to yield a true claim.)

a. Every **C/** argument is valid (regardless of conclusion type).

b. A **T/C** argument is bound to be invalid.

c. An **N/C** argument is bound to be invalid. Every valuation where the _______ is _______ is a validity counterexample.

d. A **T/N** argument is bound to be invalid. Every valuation where the _______ is _______ is a validity counterexample.

e. An **N/N** argument may be valid and may be invalid.

F. As noted, **the dual of a tautology is a contradiction, and vice versa**. On this basis, explain why the following is true.

> **If a sentence is neither a tautology nor a contradiction, its dual is neither a tautology nor a contradiction.**

(The argument could also be based on the details of the True/False Swap.)

G. We noted that the **dual of the negation of a tautology** is a tautology, as is the **negation of the dual** of a tautology; and the same holds for contradictions.[4] But since all tautologies are logically equivalent, and all contradictions are likewise logically equivalent, we could rephrase that point in terms of logical equivalence.

> A tautology is logically equivalent to the dual of its negation.
> A tautology is logically equivalent to the negation of its dual.
>
> A contradiction is logically equivalent to the dual of its negation.
> A contradiction is logically equivalent to the negation of its dual.

Does this hold for all sentences (not just tautologies and contradictions)? That is: can we say of **any** formal sentence that it's **logically equivalent to the dual of its negation**? To the **negation of its dual**?

H. A **consistent** sentence is one which isn't a contradiction. Is it true of **any** consistent sentence that its dual will also be consistent? (Is your answer consistent with the earlier claim, in Problems E and F, that **the dual of a neither-tautology-nor-contradiction is another 'neither'**?)

---

[4] In 2.34 §3.